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The brachistochrone problem in open quantum systems

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Abstract

Recently, the quantum brachistochrone problem has been discussed in the literature by using non-Hermitian Hamilton operators of different types. Here, it is demonstrated that the passage time is tunable in realistic open quantum systems due to the biorthogonality of the eigenfunctions of the non-Hermitian Hamilton operator. As an example, the numerical results obtained by Bulgakov *et al* for the transmission through microwave cavities of different shapes are analyzed from the point of view of the brachistochrone problem. The passage time is shortened in the crossover from the weak-coupling to the strong-coupling regime where the resonance states overlap and many branch points (exceptional points) in the complex plane exist. The effect can *not* be described in the framework of the standard quantum mechanics with the Hermitian Hamilton operator and consideration of S matrix poles.

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1. Introduction

Recently, the quantum brachistochrone problem has been discussed in the literature with great interest. It consists of finding the minimal time for the transition from a given initial state $|\psi_i\rangle$ to a given final state $|\psi_f\rangle$ with $|\psi_f\rangle = \mathrm{e}^{-\mathrm{i}\tau H}|\psi_i\rangle$. Bender *et al* [1] found that this minimal (passage) time can be made arbitrarily small by parametrical variation of H when H is a non-Hermitian but $\mathcal{PT}$-symmetric Hamiltonian. Assis and Fring [2] demonstrated that such a phenomenon can also be obtained for dissipative systems and concluded that the effect of a tunable passage time can be attributed to the non-Hermitian nature of the time-evolution operator rather than to its $\mathcal{PT}$-symmetry. In another paper devoted to this topic, Mostafazadeh [3] showed that it is impossible to achieve faster unitary evolutions using $\mathcal{PT}$-symmetric or other non-Hermitian Hamiltonians than those given by Hermitian Hamiltonians. Günther *et al* [4] found that the passage time is reduced under the influence of exceptional points being branch points in the complex energy plane. The quantum brachistochrone problem has been considered recently also by other authors [5–9].

The question remains open whether or not this effect is observable in a realistic quantum mechanical system. The best way to find an answer to this question is the consideration of the transmission through a quantum mechanical device such as, e.g., a microwave cavity. The waves propagate in the leads attached to the cavity which, on its part, represents an 'impurity' for the propagation. According to standard quantum mechanics, the propagation of the waves through the cavity occurs at the positions in energy of resonance states. The transmission peaks have a structure characteristic of resonances. The time which the wave spends in the system is determined by the lifetime of the resonance states. That means, the transmission occurs via so-called *standing modes*. This resonance picture of the transmission process describes well the experimentally observed situation as long as the individual (long-lived) resonance states are well isolated from one another. It breaks down, however, in the regime of strongly overlapping resonances as numerical studies on Sinai billiards of different shapes as well as on quantum billiards of Bunimovich type in the framework of the tight-binding lattice model [10] have shown [11]. In this regime, the transmission picture does not show any resonance structure. Instead, the transmission is plateau-like as a function of energy [11–13]. It is enhanced and the delay time (determined by the lifetime of the resonance states) is shortened [11–14]. Finally, the system becomes transparent and *traveling modes* appear inside the system. This behavior of the transmission probability is shown to be correlated with a reduction of the phase rigidity of the scattering wave function inside the system and with spectroscopic reordering processes taking place in it [11, 13].

It is the aim of the present paper to show that the enhanced transmission through a quantum billiard in the regime of strongly overlapping resonances as well as the shortened delay time accompanying it, can be traced back to the existence of branch points[1] in the complex plane at which the eigenvalues of at least two eigenstates coalesce. Under the influence of these points, the phases of the eigenfunctions of the non-Hermitian Hamilton operator describing the open quantum system, cease to be rigid. This behavior contrasts with the rigidity of the phases of the eigenfunctions of a Hermitian Hamilton operator. Thus, the shortening of the evolution time in physical systems whose states are described by a non-Hermitian Hamilton operator is a realistic effect, indeed, and can be observed in realistic open quantum systems in the regime of overlapping resonances.

In section 2 of the present paper, the appearance of the non-Hermitian Hamilton operator $H_{\rm eff}$ in the Feshbach projection operator (FPO) technique is sketched. In section 3, the mathematical freedom in the normalization of the eigenfunctions ϕ_λ of $H_{\rm eff}$ is discussed as well as the consequences of the chosen normalization for the rigidity of the phases of the ϕ_λ. The phase rigidity r_λ is introduced and compared with the results of an experimental study performed on a microwave cavity. In the next section (section 4), the phase rigidity ρ of the scattering wave function inside the system is defined. At the considered energy E of the system it contains, in the regime of overlapping resonances, contributions of all the individual resonance states λ including their phase rigidities r_λ. The results of the S matrix theory for the transmission through a microwave cavity are sketched in section 5. Deviations from the standard theory based on the resonance structure of the transmission appear only in the regime of overlapping resonances. Here, the phase rigidity ρ is reduced due to the many branch points characteristic of this regime. It may happen $\rho \to 0$ in a broader parameter range. In such a case, the transmission has a plateau-like structure and may occur via traveling modes. The

[1] In the mathematical literature, the branch points are often called *exceptional points*. Their physical meaning is based upon their topological structure and their relation to, respectively, level repulsion and widths bifurcation occurring in approaching them under different conditions. For details see, e.g., the reviews [15]. In the present paper, the concept *branch point* is preferred since the physical properties of resonance states in the neighborhood of these points (such as widths bifurcation) play the decisive role.

system becomes transparent. In section 6, the results are summarized.and some conclusions are drawn.

2. Feshbach projection operator (FPO) technique

In the present paper, the FPO technique [16] will be used in order to describe the transmission through an open quantum microwave cavity. In the FPO formalism, the full function space is divided into two subspaces: the Q subspace contains all wave functions that are localized inside the system and vanish outside of it while the wave functions of the P subspace are extended up to infinity and vanish inside the system, see [15]. It is $P + Q = 1$. In this formalism, two Hamilton operators characterize the system. The first one, H, is Hermitian. It describes the scattering in the whole function space,

$$(H - E)\Psi_C^E = 0, \tag{1}$$

consisting of the two subspaces: the subspace of discrete states of the considered (closed) system (described by the Hermitian operator H_B) and of the subspace of scattering states (continuum described by the Hermitian operator H_C) into which the system is embedded. In solving (1) in the whole function space by using the FPO technique [16], the effective non-Hermitian Hamilton operator

$$H_{\text{eff}} = H_B + \sum_C V_{BC} \frac{1}{E^+ - H_C} V_{CB} \tag{2}$$

appears which contains H_B as well as an additional symmetrical non-Hermitian term that describes the coupling of the resonance states via the common environment. Here V_{BC}, V_{CB} stand for the coupling matrix elements between the *eigenstates* of H_B and the environment [15] that may consist of different continua C. The operator H_{eff} is non-Hermitian,

$$(H_{\text{eff}} - z_\lambda)\phi_\lambda = 0, \tag{3}$$

its eigenvalues z_λ and eigenfunctions ϕ_λ are complex. The eigenvalues provide not only the energies of the resonance states but also their widths (inverse lifetimes). The eigenfunctions are biorthogonal.

The eigenvalues and eigenfunctions of H_B contain the interaction u of the discrete states which is given by the nondiagonal matrix elements of H_B. This interaction is of standard type in closed systems and may be called therefore internal interaction. The eigenvalues and eigenfunctions of H_{eff} contain additionally the interaction v of the resonance states via the common continuum (v is used here instead of the concrete matrix elements of the second term of H_{eff}). This part of interaction is, formally, of second order and may be called external interaction. While u and $\text{Re}(v)$ cause level repulsion in energy, $\text{Im}(v)$ is responsible for the bifurcation of the widths of the resonance states (resonance trapping). The phenomenon of resonance trapping appearing in the regime of overlapping resonances, has been proven experimentally in a microwave cavity [17].

Since the effective Hamilton operator (2) depends explicitly on the energy E, so do its eigenvalues z_λ and eigenfunctions ϕ_λ. Far from thresholds, the energy dependence is weak, as a rule, in an energy interval of the order of magnitude of the width of the resonance state. The solutions of the fixed-point equations $E_\lambda = \text{Re}\,(z_\lambda)_{|E=E_\lambda}$ and of $\Gamma_\lambda = -2\text{Im}(z_\lambda)_{|E=E_\lambda}$ are numbers that coincide with the poles of the S matrix. The widths Γ_λ determine the time scale characteristic of the resonance states λ. Using the FPO formalism with the non-Hermitian Hamilton operator H_{eff}, it is however not necessary to look for the poles of the S matrix since the spectroscopic information is involved in the complex eigenvalues z_λ of H_{eff}. In the

S matrix, the eigenvalues z_λ with their full energy dependence appear. Due to this fact, the S matrix contains information on the environment of the considered resonance states such as the position of decay thresholds and of neighboring resonance states.

Thus, the FPO formalism may be considered as an extension [18] of the R matrix theory used in standard quantum mechanics for the description of decaying states. The standard spectroscopic parameters (positions, widths and partial widths of the resonance states λ) are replaced by the energy-dependent functions $E_\lambda, \Gamma_\lambda$ and coupling matrix elements between system and environment. While R matrix theory gives reasonable results only for narrow non-overlapping resonance states, the FPO formalism can be used for all resonance states including the broad ones in the overlapping regime. The influence of neighboring resonances as well as of decay thresholds is taken into account via the energy dependence of the eigenvalues z_λ and eigenfunctions ϕ_λ. The spectroscopic information can be controlled by means of an external parameter. Also the redistribution processes taking place under the influence of the coupling to the continuum in the overlapping regime can be traced. The results obtained in the FPO formalism pass into those of the R matrix theory when the overlapping of the resonance states vanishes.

3. Phase rigidity of the eigenfunctions ϕ_λ of the non-Hermitian Hamilton operator $H_{\rm eff}$

The eigenfunctions ϕ_λ of the non-Hermitian symmetrical Hamilton operator $H_{\rm eff}$ are complex and biorthogonal. The normalization condition $\langle \phi_\lambda^{\rm left} | \phi_\lambda^{\rm right} \rangle = \langle \phi_\lambda^* | \phi_\lambda \rangle$ fixes only two of the four free parameters [4]. This freedom can be used in order to provide a smooth transition from an open quantum system (with, in general, nonvanishing decay widthsΓ_λ of its states and biorthogonal wave functions ϕ_λ) to the corresponding closed one (with $\Gamma_\lambda \to 0$ and real wave functions that are normalized in the standard manner): $\langle \phi_\lambda^* | \phi_\lambda \rangle \to \langle \phi_\lambda | \phi_\lambda \rangle = 1$ if the coupling vectors in the non-Hermitian part of (2) vanish. That means, the orthonormality conditions can be chosen as

$$\langle \phi_\lambda^* | \phi_{\lambda'} \rangle = \delta_{\lambda,\lambda'} \tag{4}$$

with the consequence that [15]

$$\langle \phi_\lambda | \phi_\lambda \rangle \equiv A_\lambda \geqslant 1 \tag{5}$$

$$B_\lambda^{\lambda'} \equiv \langle \phi_\lambda | \phi_{\lambda' \neq \lambda} \rangle = -B_{\lambda'}^{\lambda} \equiv -\langle \phi_{\lambda' \neq \lambda} | \phi_\lambda \rangle \qquad \left| B_\lambda^{\lambda'} \right| \geqslant 0. \tag{6}$$

The normalization condition (4) entails that the phases of the eigenfunctions in the overlapping regime are not rigid: the normalization condition $\langle \phi_\lambda^* | \phi_\lambda \rangle = 1$ is fulfilled only when $\mathrm{Im}\langle \phi_\lambda^* | \phi_\lambda \rangle \propto \mathrm{Re}\phi_\lambda \cdot \mathrm{Im}\phi_\lambda = 0$, i.e. by rotating the wave function at a certain angle β_λ. The phases of the wave functions of the original states corresponding to $v = 0$ (vanishing non-diagonal matrix elements of the second term of (2)) are fixed, say to $\beta_\lambda^0 = 0$ or $\pm\pi$, so that $\mathrm{Im}\phi_\lambda^0 = 0$. The influence of a neighboring state is described by $v \neq 0$ (i.e., by the non-diagonal matrix elements of the second term of (2)). At $v \neq 0$, the angle β_λ is different from β_λ^0, generally. The difference $\left| \beta_\lambda - \beta_\lambda^0 \right|$ may be $\pm\pi/4$ at most, corresponding to $\mathrm{Re}\phi_\lambda = \pm\,\mathrm{Im}\,\phi_\lambda$ (as compared to $\mathrm{Im}\phi_R^0 = 0$). This maximum value appears at a branch point in the complex energy plane (see footnote 1) where two eigenvalues of $H_{\rm eff}$ coalesce [4, 12, 15]. Here [4, 15, 19]

$$\phi_\lambda \to \pm \mathrm{i}\phi_{\lambda'}, \qquad \phi_{\lambda'} \to \mp \mathrm{i}\phi_\lambda. \tag{7}$$

The phase rigidity defined by

$$r_\lambda = \frac{\langle \phi_\lambda^* | \phi_\lambda \rangle}{\langle \phi_\lambda | \phi_\lambda \rangle} = \frac{1}{(\mathrm{Re}\,\phi_\lambda)^2 + (\mathrm{Im}\,\phi_\lambda)^2} = \frac{1}{A_\lambda} \tag{8}$$

is a useful measure [13] for the rotation angle β_λ. When the resonance states are distant from one another, it is $r_\lambda \approx 1$ due to $\langle\phi_\lambda|\phi_\lambda\rangle \approx \langle\phi_\lambda^*|\phi_\lambda\rangle$. In approaching a branch point in the complex energy plane [15, 19], we have $\langle\phi_\lambda|\phi_\lambda\rangle \equiv A_\lambda \to \infty$ and $r_\lambda \to 0$. Therefore $1 \geqslant r_\lambda \geqslant 0$.

It should be underlined that, after defining the normalization condition (4), the values r_λ are fixed by the coupling matrix elements v of H_{eff} characteristic of the degree of overlapping of the resonance states. They can be varied by controlling the system by means of external parameters, e.g. by means of a laser in the case of an atom with many levels (for concrete examples see [20]). The rotation angle β_λ as well as the values A_λ and r_λ may be considered to be a synonym for the biorthogonality of the eigenfunctions ϕ_λ of the non-Hermitian Hamiltonian (2). They are a measure for the distance of the considered states from a branch point in the complex plane and for the spectroscopic reordering processes occurring in an open quantum system with overlapping resonance states under the influence of the coupling to the continuum. Physically, the phase rigidity r_λ measures the degree of alignment of one of the neighboring resonance states with one of the scattering states ξ_C^E of the environment. This alignment takes place at the cost of the other states that decouple, to a certain extent, from the environment (*widths bifurcation* or *resonance trapping* occurring in the neighborhood of a branch point in the complex energy plane [15]). The r_λ are, generally, different for the different states λ.

We consider now the experimental results obtained on a microwave cavity [21]. The experimental conditions are chosen in such a manner that the phase difference between the oscillating fields at the position of the antennas is π far from the branch point (see footnote 1). Then the phase difference is traced experimentally in approaching the branch point: in a comparably large parameter range, it drops eventually to $\pi/2$ at the branch point. For an interpretation of the results, the authors [21] consider the reduced phase difference only at the branch point and relate it to the existence of a chiral state. They do not discuss the smooth reduction from π to almost $\pi/2$ in approaching the branch point.

According to the discussion above, the experimentally observed [21] reduction of the phase difference between the wave functions of the two states can be related to the reduction of the phase rigidity of the two wave functions. The phase rigidity drops smoothly from its maximum value $r_\pm = 1$ far from the branch point (with the phase difference π (or 2π) between the wave functions of isolated resonance states) to its minimum value $r_\pm = 0$ at the branch point (with the phase difference $\pm\pi/2$ according to (7)). This interpretation explains, in a natural manner, the experimentally observed smooth reduction of the phase difference in a comparably large parameter range. Also the phase jump occurring at the branch point [4] is not in disagreement with the experimental data. The results demonstrate the (parametric) dynamics of open quantum systems which is generated by the interaction of resonance states via the continuum as discussed above.

4. Phase rigidity of the scattering wave function Ψ_C^E

The solution of the whole problem (1) with the Hermitian Hamilton operator H reads [15]

$$\Psi_C^E = \xi_C^E + \sum_\lambda \Omega_\lambda^C \frac{\langle\phi_\lambda^*|V|\xi_C^E\rangle}{E - z_\lambda}, \tag{9}$$

where

$$\Omega_\lambda^C = \left(1 + \frac{1}{E^+ - H_C} V_{CB}\right)\phi_\lambda \tag{10}$$

is the wave function of the resonance state λ and the ξ_C^E are the (coupled) scattering wave functions of the continuum into which the system is embedded. According to (9), the eigenfunctions ϕ_λ of the non-Hermitian Hamilton operator $H_{\rm eff}$ give the main contribution to the scattering wave function $\hat{\Psi}_C^E$ in the interior of the system,

$$\Psi_C^E \to \hat{\Psi}_C^E = \sum_\lambda c_{\lambda E}\phi_\lambda, \qquad c_{\lambda E} = \frac{\langle \phi_\lambda^* | V | \xi_C^E \rangle}{E - z_\lambda}. \tag{11}$$

The weight factors $c_{\lambda E}$ contain the excitation probability of the states λ.

In the FPO method supplemented by the normalization condition (4), the definition of the two subspaces (system and environment) appears in a natural manner: H_B describes the closed system which becomes open when embedded in the continuum of scattering wave functions ξ_C^E described by H_C. Therefore, all spectroscopic values characteristic of resonance states can be traced to the corresponding values of discrete states by controlling the coupling to the continuum. That means, with $v \to 0$, the transition from resonance states (described by the non-Hermitian $H_{\rm eff}$) to discrete states (described by the Hermitian H_B) can be controlled.

Let us consider the one-channel case, $C = 1$, and $\Psi_C^E \to \hat{\Psi}^E$ in the interior of the system. From (11) follows for the right and left wave functions

$$|\hat{\Psi}^{E,R}\rangle = \sum_\lambda c_{\lambda E} \left|\phi_\lambda^R\right\rangle \tag{12}$$

$$\langle\hat{\Psi}^{E,L}| = \sum_\lambda d_{\lambda E} \left\langle\phi_\lambda^L\right| \tag{13}$$

with $\left|\phi_\lambda^R\right\rangle \equiv |\phi_\lambda\rangle$, $\left\langle\phi_\lambda^L\right| = \langle\phi_\lambda^*|$ and $d_{\lambda E} = c_{\lambda E}^*$ when excitation and decay of the state λ occur via the same mechanism. Therefore the $\hat{\Psi}^E$ can be normalized,

$$\begin{aligned}\langle\hat{\Psi}^{E,L}|\hat{\Psi}^{E,R}\rangle &= \sum_{\lambda\lambda'} c_{\lambda E}^* c_{\lambda' E}\langle\phi_\lambda^*|\phi_{\lambda'}\rangle \\ &= \sum_\lambda |c_{\lambda E}|^2 \equiv 1.\end{aligned} \tag{14}$$

The normalization has to be done separately at every energy E due to the explicit energy dependence of the $c_{\lambda E}$. Moreover,

$$\begin{aligned}\langle\hat{\Psi}^{E,L*}|\hat{\Psi}^{E,R}\rangle &= \sum_{\lambda\lambda'} c_{\lambda E} c_{\lambda' E}\langle\phi_\lambda|\phi_{\lambda'}\rangle \\ &= \sum_\lambda (c_{\lambda E})^2 A_\lambda + \sum_{\lambda<\lambda'} c_{\lambda E} c_{\lambda' E}\left(B_\lambda^{\lambda'} + B_{\lambda'}^{\lambda}\right) \\ &= \sum_\lambda (c_{\lambda E})^2 A_\lambda\end{aligned} \tag{15}$$

due to $B_\lambda^{\lambda'} = -B_{\lambda'}^{\lambda}$, see (6). A_λ is a real number, see [15]. From (14) and (15) follows

$$\frac{\langle\hat{\Psi}^{E*}|\hat{\Psi}^E\rangle}{\langle\hat{\Psi}^E|\hat{\Psi}^E\rangle} = \sum_\lambda (c_{\lambda E})^2 A_\lambda = \sum_\lambda \frac{(c_{\lambda E})^2}{r_\lambda}, \tag{16}$$

and the phase rigidity ρ of the wavefunctions $\hat{\Psi}^E$ may be defined by

$$\begin{aligned}\rho &= \mathrm{e}^{2\mathrm{i}\theta} \sum_\lambda \frac{\mathrm{Re}\,[(c_{\lambda E})^2]}{r_\lambda} \\ &= \mathrm{e}^{2i\theta} \sum_\lambda \frac{1}{r_\lambda}([\mathrm{Re}(c_{\lambda E})]^2 - [\mathrm{Im}(c_{\lambda E})]^2)\end{aligned} \tag{17}$$

in analogy to (8). The value ρ corresponds to a rotation of $\hat{\Psi}^E$ by θ corresponding to the ratio between its real and imaginary parts. In spite of the complicated structure of ρ, it holds $1 \geqslant \rho \geqslant 0$ (since $1 \leqslant (a^2 - b^2)/(a^2 + b^2) \leqslant 0$ for every summand $(a + \mathrm{i}b)^2$ in (17)). Equations (15) and (17) show that the definition of ρ is meaningful only when the sum of all the overlapping states λ at the energy E of the system is considered. The value ρ is uniquely determined by the spectroscopic properties of the system that are expressed by the coupling coefficients to the environment and the level density, or by the positions and widths of the resonance states and the phase rigidities r_λ.

According to (17), we have the following border cases.

(i) The resonances are well separated from one another, $\Gamma_\lambda \ll \Delta E \equiv E_\lambda - E_{\lambda'} : r_\lambda \approx 1$ and $(c_{\lambda E})^2 \approx |c_{\lambda E}|^2 = 1$ for $E \to E_\lambda$. In such a case $|\rho| \to 1$.

(ii) The resonances overlap and $r_\lambda < 1$ (but different from 0) for a certain number of neighboring resonances: it may happen that $\rho = 0$ in a finite energy interval, see [11–13] for numerical examples.

(iii) The eigenvalues z_λ of two resonance states coalesce at $E \to E_\lambda$: $r_\lambda \to 0$ and $(c_{\lambda E})^2 \to 0$ at this energy, see e.g. [15]. Therefore ρ is finite at $E \to E_\lambda$. The results of a numerical example (double quantum dot) are shown in [13], figure 2.

(iv) K out of N wave functions Ψ_C^E are aligned with the K scattering wave functions ξ_C^E of the environment while the remaining $N - K$ wave functions are more or less decoupled from the continuum and well separated from one another. In such a case, $|\rho| \to 1$. In contrast to the first case, the $N - K$ trapped (narrow) resonance states are superposed by a background term that arises from the K aligned (short-lived) resonance states.

This behavior of the phase rigidity ρ is traced in a numerical study for different quantum billiards [11]. When the beam is fully reflected, it may, of course, also happen that $|\rho| \to 1$ in a finite energy interval.

The wave functions $\hat{\Psi}^E$ are the exact solutions of the Schrödinger equation (1) in the interior of the system. Equation (17) shows that ρ obtained for these wave functions is related to the individual r_λ. This relation becomes important only in the regime of overlapping resonances where $r_\lambda < 1$. Every value r_λ as well as every coefficient $c_{\lambda E}$ are given by the concrete values of the coupling strength between system and environment in the concrete situation considered. Thus, also ρ is uniquely determined by the degree of overlapping of the resonance states by which the coupling matrix elements are determined.

This result is in contrast to the definition of ρ_{br} given by Brouwer [22] by means of an arbitrary wave function $\tilde{\Psi}$ although

$$\rho_{\mathrm{br}} = \mathrm{e}^{2\mathrm{i}\Theta} \frac{\int \mathrm{d}r (|\mathrm{Re}\,\tilde{\Psi}(r)|^2 - |\mathrm{Im}\,\tilde{\Psi}(r)|^2)}{\int \mathrm{d}r (|\mathrm{Re}\,\tilde{\Psi}(r)|^2 + |\mathrm{Im}\,\tilde{\Psi}(r)|^2)} \tag{18}$$

is formally analog to the definition (17). In the case of ρ_{br}, the source for the reduction of the phase rigidity is not known. It is rather expressed quite generally by the value ρ_{br} in analyzing experimental data. Unlike ρ_{br}, the only source for the reduced phase rigidity (17) is the biorthogonality of the eigenfunctions ϕ_λ of the non-Hermitian Hamilton operator H_{eff} by which the alignment of individual wave functions ϕ_λ with the scattering wave functions ξ_C^E of the environment becomes possible. It can be calculated as shown, e.g., in [11, 13]. The alignment may be characterized by the corresponding rotation angles β_λ or the phase rigidities r_λ as discussed in section 3. This effect is characteristic of open quantum systems in the regime of overlapping resonances. It appears also at zero temperature.

5. Transmission through a microwave cavity

5.1. Isolated resonances

According to the S matrix theory, the amplitude for the transmission through a quantum dot is [23]

$$t = -2\pi\,\mathrm{i}\sum_{\lambda}\frac{\langle\xi_L^E|V|\phi_\lambda\rangle\langle\phi_\lambda^*|V|\xi_R^E\rangle}{E-z_\lambda}. \tag{19}$$

The eigenvalues z_λ and eigenfunctions ϕ_λ of $H_{\rm eff}$ are involved in (19) with their full energy dependence.

For $\rho = 1$ and well isolated resonance states, the transmission amplitude (19) repeats the resonance structure of (9) of the wave function Ψ_C^E. The transmission peaks appear at the positions $E_\lambda \equiv \mathrm{Re}\,(z_\lambda)_{|E=E_\lambda} \approx E_\lambda^B$ of the resonance states. Using the relation [15]

$$\begin{aligned}\Gamma_\lambda &= 2\pi\left\{\langle\xi_L^E|V|\phi_\lambda\rangle\langle\phi_\lambda^*|V|\xi_L^E\rangle + \langle\xi_R^E|V|\phi_\lambda\rangle\langle\phi_\lambda^*|V|\xi_R^E\rangle\right\}\\ &= 4\pi\langle\xi_C^E|V|\phi_\lambda\rangle\langle\phi_\lambda^*|V|\xi_C^E\rangle\end{aligned} \tag{20}$$

for the case of a symmetrical cavity with isolated resonance states and one channel in each of the two identical (semi-infinite) leads ($C = L, R$, respectively), the peak height is

$$|t_{(E\to E_\lambda)}| = \frac{4\pi}{\Gamma_\lambda}\left|\langle\xi_L^E|V|\phi_\lambda\rangle\langle\phi_\lambda^*|V|\xi_R^E\rangle\right| = 1. \tag{21}$$

Except for threshold effects, the profile of the transmission peak is of Breit-Wigner type, determined by the width $\Gamma_\lambda \equiv -2\mathrm{Im}\,(z_\lambda)_{|E=E_\lambda}$ of the resonance state λ.

An analogous result holds when there is a nonvanishing background term additional to the resonance term (19) of the transmission amplitude. Such a term is caused by the contribution of the scattering wave functions ξ_C^E in (9) to the transmission. It describes a wave traveling through the cavity. The time scale corresponding to this so-called *direct* part of the transmission is, generally, well separated from that corresponding to the resonance part described by (19).[2] Mostly, the resonances are narrow and well separated from one another. They appear as Fano resonances [24] on the smooth background (caused by the traveling mode ξ_C^E). Due to the different time scales of the resonance and direct processes, it is $|\rho| \approx 1$ also in this case.

Thus, the resonance structure of the transmission amplitude with and without contributions from the direct reaction part can be described in the framework of standard quantum mechanics with the Hermitian Hamilton operator and consideration of S matrix poles since the phases of the wave functions of the resonance states are almost rigid, $\rho \approx 1$.

5.2. Overlapping resonances

The situation is another one when the resonances overlap. In the overlapping regime, the resonance states avoid crossings with neighbored resonance states. In contrast to (20), it holds

$$\Gamma_\lambda < 4\pi\langle\xi_C^E|V|\phi_\lambda\rangle\langle\phi_\lambda^*|V|\xi_C^E\rangle \tag{22}$$

in the case with one channel in each of the two identical leads due to the biorthogonality of the eigenfunctions ϕ_λ [15]. At $E \to E_\lambda$, the transmission amplitude is

$$t_{(E\to E_\lambda)} = -2\pi\,\mathrm{i}\sum_{\lambda'\neq\lambda}\frac{\langle\xi_L^E|V|\phi_{\lambda'}\rangle\langle\phi_{\lambda'}^*|V|\xi_R^E\rangle}{E-z_{\lambda'}} - 4\pi\frac{\langle\xi_L^E|V|\phi_\lambda\rangle\langle\phi_\lambda^*|V|\xi_R^E\rangle}{\Gamma_\lambda}. \tag{23}$$

[2] In many-body quantum systems such as nuclei, this fact is basic for the *Unified Theory of Nuclear Reactions* [16] developed by Feshbach. Here, the (long-lived) states of the Q subspace are not calculated explicitly (as in [15] and, e.g., in [23]), but are described by using statistical assumptions.

It follows from (22) that the contribution of the state λ to $t_{(E\to E_\lambda)}$ is larger than 1. The unitarity condition will be fulfilled, nevertheless, due to the possibility of rotating the ϕ_λ, i.e. due to phase changes of the wave functions ϕ_λ. Moreover, also the minima in the transmission between two resonance peaks may be filled up due to phase changes of the wave functions ϕ_λ and $\phi_{\lambda'}$ of the two neighboring resonance states λ and λ' [15]. As a consequence, the transmission in the overlapping regime does not show a resonance structure. Instead, it might be nearly plateau-like. Let us rewrite therefore the transmission amplitude (19) by means of the scattering wave function (11),

$$t = -2\pi \mathrm{i}\langle \xi_L^E | V | \hat{\Psi}_R^E \rangle \tag{24}$$

with $\hat{\Psi}_R^E$ being complex, in general. The advantage of this representation consists of the fact that it does not suggest the existence of resonance peaks in the transmission probability. Quite the contrary, the transmission is determined by the degree of alignment of the wave function $\hat{\Psi}_C^E$ with the propagating modes ξ_C^E in the leads, i.e. by the value $\langle \xi_C^E | V | \hat{\Psi}_C^E \rangle$. Nevertheless, the expressions (24) and (19) are fully equivalent.

The plateau-like structure of the transmission cannot be obtained in standard quantum mechanics with fixed phases of the wave functions, $r_\lambda = 1$ and $\rho = 1$. It is generated by the interference processes with account of the alignment of some of the resonance states to the scattering states ξ_C^E of the environment. At most, $\mathrm{Re}\,\hat{\Psi}_C^E = \pm \mathrm{Im}\,\hat{\Psi}_C^E$ (as for the ξ_C^E). This case corresponds to $\rho = 0$. It will be reached when many resonance states are almost aligned with the ξ_C^E and $\sum_\lambda \mathrm{Re}\,[(c_{\lambda E})^2]/r_\lambda \approx 0$ according to (17).

Let us now consider the case of two resonance states with extremely strong overlapping (corresponding to $r_{\lambda_1} = r_{\lambda_2} = 0$) which occurs at the branch point in the complex energy plane. Here two eigenvalues z_1 and z_2 of H_{eff} coalesce, $E_{\lambda_1} = E_{\lambda_2} \equiv E_\lambda$, $\Gamma_{\lambda_1} = \Gamma_{\lambda_2} \equiv \Gamma_\lambda$. In the case of one channel in each of the two identical leads, it follows from (19)

$$t_{(E\to E_\lambda)} \to \frac{4\pi}{\Gamma_\lambda}\left(\langle \xi_L^E | V | \phi_{\lambda_1}\rangle\langle \phi_{\lambda_1}^* | V | \xi_R^E\rangle + \langle \xi_L^E | V | \phi_{\lambda_2}\rangle\langle \phi_{\lambda_2}^* | V | \xi_R^E\rangle\right) = 0 \tag{25}$$

at $E \to E_\lambda$ due to $|\phi_\lambda\rangle \to \pm \mathrm{i}|\phi_{\lambda'\neq\lambda}\rangle$ at the branch point, equation (7). That means, the transmission vanishes at the energy $E = E_\lambda$ of the two resonance states. The transmission profile can be derived from (19) by taking into account the resonance behavior of the coupling coefficients of the two resonance states [25, 26],

$$t = -2\mathrm{i}\frac{\Gamma_\lambda}{E - E_\lambda + \frac{\mathrm{i}}{2}\Gamma_\lambda} - \left(\frac{\Gamma_\lambda}{E - E_\lambda + \frac{\mathrm{i}}{2}\Gamma_\lambda}\right)^2. \tag{26}$$

The interference between both terms in (26) causes two transmission peaks in an energy region ΔE that is characteristic of the first term of (26). The resulting 'antiresonance' at $E = E_\lambda$ is narrower than a Breit-Wigner resonance, and the two transmission peaks are non-symmetrical. Let us compare the transmission in the energy region ΔE when (i) there are two coalesced eigenvalues of H_{eff} as discussed above and (ii) there are two (more or less) isolated resonance states resulting in two symmetrical transmission peaks of Breit–Wigner shape. In both cases we have two transmission peaks, however with a different profile. As a consequence, the transmission is different in the two cases. It is larger in the first case than in the second one. The difference is however small.

Thus, $\rho \neq 0$ in accordance with (17) in spite of $r_\lambda = 0$ for the two states whose eigenvalues coalesce at the branch point in the complex plane. Due to the reduced phase rigidities r_λ of the two states λ_1 and λ_2, this case can be described by standard quantum mechanics with Hermitian Hamilton operators and fixed phases of its states at the best in an approximate manner.

5.3. Relation to the phase rigidity ρ

As a result of the above discussion, we have the following cases.

(1) The phases of the eigenfunctions ϕ_λ of the non-Hermitian Hamilton operator $H_{\rm eff}$ are (almost) rigid, $|r_\lambda| \approx 1$ and $|\rho| \approx 1$.

In this case, the transmission can be described quite well by standard quantum mechanics with a Hermitian Hamilton operator and fixed phases. The transmission shows a resonance structure according to the standing waves in the cavity. The time delay of the transmission inside the cavity is caused by the finite lifetime of the individual resonance states.

(2) The phases of the eigenfunctions ϕ_λ of the non-Hermitian Hamilton operator $H_{\rm eff}$ are not rigid, $|r_\lambda| < 1$ and $|\rho| < 1$.

In this case, the transmission cannot be described by standard quantum mechanics with a Hermitian Hamilton operator, and the transmission does not show any pronounced resonance structure. In a comparably large parameter range, it is rather plateau-like and the transmission occurs, in this parameter range, via traveling modes through the cavity, i.e. the cavity becomes transparent. The transmission does not occur through individual resonance states in this case. Instead, the overlapping of the resonance states allows the alignment of some of them with the traveling (scattering) states of the environment so that the cavity does not cause a time delay of the transmission.

The numerical results [11] obtained for the transmission through microwave cavities of different shapes show exactly the features discussed above. In the weak-coupling regime as well as in the strong-coupling regime, the transmission shows a resonance structure as expected from the standard quantum mechanics with a Hermitian Hamilton operator. The only difference between the two cases is the appearance of a smooth background term in the strong-coupling regime which does not exist in the weak-coupling case, and the reduction of the number of resonance peaks by two (corresponding to the alignment of two resonance states each with one channel in each of the two identical attached leads).

In the crossover from the weak-coupling regime to the strong-coupling one, however, the transmission is plateau-like instead of showing a resonance structure. It is enhanced as compared to the transmission probability in the two borderline cases. In this regime, the resonance states overlap and spectroscopic reordering processes take place. Due to widths bifurcation, some of the resonance states become short lived while other ones become trapped (long-lived). The enhancement of the transmission is caused by the short-lived states. Most interesting is the correlation between transmission $|t|$ and reduced phase rigidity $1 - |\rho|$ which can be seen very clearly in all the numerical results shown in [11]. The transmission in the crossover regime is not only enhanced but it also outspeeds the transmission calculated in standard quantum mechanics. The reason is the formation of aligned (short-lived) resonance states in the neighborhood of branch points in the complex plane.

The behavior of the transmission in the crossover regime with overlapping resonance states does *not* correspond to the expectations of the standard quantum mechanics with the Hermitian Hamilton operator, rigid phases of its eigenfunctions, and decay widths obtained from poles of the S matrix. This can be seen also in the following manner. The time, the wave spends inside the system at the energy of a resonance state, can be expressed by the Wigner time delay function which is proportional to the width of the state. Numerical calculations performed for a quantum billiard with overlapping resonance states by using the non-Hermitian Hamilton operator (2) show that the spectroscopic redistribution processes can be seen, indeed, also in the time delay function [14]. There is almost no time delay in the energy range of a short-lived state while it is large at the energies of the trapped states. In the standard quantum

mechanics with a Hermitian Hamilton operator, the spectroscopic redistribution processes are not involved. Therefore, short-lived (aligned) resonance states do not appear, the delay time cannot be reduced and the transmission time cannot be shortened.

6. Conclusions

The quantum brachistochrone problem of a physical system can be studied by considering the time needed for the transmission through the system from one of the attached leads to another one. According to S matrix theory, the transmission time at a certain energy E is determined by the lifetime of the resonance states lying at this energy. The lifetime of a resonance state is bounded from below: it cannot be smaller than allowing traveling through the system in accordance with traveling through the attached leads, i.e. the system may become transparent at most[3]. The difference between Hermitian and non-Hermitian quantum systems is that this lower bound can be reached in non-Hermitian systems by aligning the wave functions of the system with those of the environment while such a possibility does not exist in Hermitian systems.

The condition that an alignment of wave functions of the system with those of the environment becomes possible in non-Hermitian quantum mechanics, is resonance overlapping such that many branch points exist in the parameter range considered. Only in the neighborhood of these branch points, the eigenfunctions ϕ_λ of the non-Hermitian Hamilton operator $H_{\rm eff}$ are really biorthogonal and have the possibility to align with the traveling waves ξ_C^E in the attached leads due to their interaction via the continuum. Mathematically, the alignment is a consequence of the normalization of the eigenfunctions of the non-Hermitian Hamiltonian $H_{\rm eff}$ according to (4). The alignment takes place in a hierarchical manner [15]. It is maximal when many levels are almost aligned. In this case, $\rho \approx 0$ in a certain range of the considered parameter and the transmission is plateau-like with $|t| \approx 1$ (for numerical examples see [11, 13]). The system is transparent, up to some dips that appear in the case when the system has many levels. These dips are caused by the long-lived trapped resonance states that always appear together with the short-lived aligned resonance states (due to widths bifurcation) in the neighborhood of the branch points. An example are the whispering gallery modes in quantum billiards of Bunimovich type [11, 27]. At the critical point at which the number of aligned states is exactly equal to the number of traveling waves ξ_C^E in the leads, $|\rho| > 0$ and $|t| < 1$.

This freedom to align the wave functions of the individual states with the traveling waves ξ_C^E in the attached leads does not exist in the Hermitian quantum mechanics. Instead, the normalization of the wave functions according to $\langle\phi_\lambda|\phi_\lambda\rangle = 1$ fixes the phases of the individual wave functions in the Hermitian quantum mechanics and prevents any alignment. As a consequence, it is always $|r_\lambda| = 1$ and $|\rho| = 1$ in standard Hermitian quantum mechanics. The transmission takes place via waves standing at a certain energy for a certain time inside the system. This time is longer than the traveling time, generally. The transmission shows a characteristic resonance structure that is described, in the standard theory, by means of the poles of the S matrix. This resonance picture can be seen in the regime of weak coupling between system and environment where the individual resonances are well isolated from one another as well as in the regime of strong coupling where narrow resonances are superposed by a smooth background. The crossover between these two borderline cases can *not* be described by standard Hermitian quantum mechanics as it is very well known in the physics of open

[3] In a many-particle system, the life time of a resonance state cannot be smaller than the lifetime of a single-particle resonance with the corresponding quantum numbers.

quantum systems. For example, an interpolation procedure between these two limiting cases is proposed in [28]. The reason for the failure of the Hermitian quantum mechanics in this case is, as shown above, $|r_\lambda| < 1$ and $|\rho| < 1$ in the crossover regime.

Summarizing it can be stated that the brachistochrone problem is observable in realistic open quantum mechanical systems. It is directly related to the branch points of the non-Hermitian Hamilton operator. In the present paper, the transmission through a small open quantum billiard is considered as an example. The system becomes transparent in the regime of overlapping resonances since the Hamilton operator $H_{\rm eff}$ is really non-Hermitian in this regime and many branch points exist in the complex plane.

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